Rotating Yang-Mills dyons in anti-de Sitter spacetime

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Abstract

We construct new axially symmetric solutions of SU(2) Yang-Mills theory in a four dimensional anti-de Sitter spacetime. Possessing nonvanishing nonabelian charges, these regular configurations have also a nonzero angular momentum. Numerical arguments are also presented for the existence of rotating solutions of Einstein-Yang-Mills equations in an asymptotically anti-de Sitter spacetime.

Introduction.– Given its important role in string theory, anti-de Sitter (AdS) spacetime has recently attracted huge interest. As proven by some authors [1, 2], even the simple spherically symmetric $SU(2)$ Einstein-Yang-Mills (EYM) system with a negative cosmological constant Λ presents some surprising results. A variety of well known features of asymptotically flat self-gravitating nonabelian solutions are not shared by their AdS counterparts. First, there is a continuum of regular and black hole solutions in terms of the adjustable shooting parameters that specifies the initial conditions at the origin or at the event horizon, rather then discrete points. The spectrum has a finite number of continuous branches. Secondly, there are nontrivial solutions stable against spherically symmetric linear perturbations, corresponding to stable monopole and dyon configurations. The solutions are classified by non-Abelian electric and magnetic charges and the ADM mass. When the parameter Λ approaches zero, an already-existing branch of monopoles and dyon solutions collapses to a single point in the moduli space [3]. At the same time new branches of solutions emerge.

An interesting physical question is whether the known static nonabelian solutions can be generalized to include an angular momentum. In a EYM theory without a cosmological constant, slowly rotating charged solutions are known to exist [4]. For regular configurations, the angular momentum and the electric charge are related. The existence of these perturbative rotating solutions came as a surprise, given the experience with other solitonic solutions [5]. However, it is not clear whether they can be extended to exact solutions [6, 7]. Thus it may be important to consider other types of asymptotics for a better understanding of this question.

Here we consider this problem in a AdS geometry for a $SU(2)$ field. Although an analytic or approximate solution still appears to be intractable, we present numerical arguments that a negative cosmological constant could support finite energy nonabelian configurations with a nonvanishing angular momentum. These regular solutions are, in nature, nontopological solitons and carry both angular momentum and electric charge. Moreover, they are found to survive in the presence of gravity.

Axially symmetric ansatz and general relations.– We consider the Yang-Mills (YM) equations

$$\nabla_\mu F^{\mu\nu} + ie[A_\mu, F^{\mu\nu}] = 0, \tag{1}$$

in a fixed AdS background

$$ds^2 = \frac{dr^2}{1-\frac{\Lambda}{3}r^2} + r^2(d\theta^2 + \sin^2\theta d\phi)^2 - (1-\frac{\Lambda}{3}r^2)dt^2, \tag{2}$$

where e is the gauge coupling constant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie\,[A_\mu, A_\nu]$ and $A_\mu = \frac{1}{2}\tau^a A_\mu^a$. We are interested in stationary, axially symmetric solutions of the $SU(2)$ YM configurations with a nonvanishing total angular momentum.

For the time translation symmetry, we choose a natural gauge such that $\partial A/\partial t$=0. However, a rotation around the $z-$axis can be compensated by a gauge rotation $\mathcal{L}_\varphi A = D\Psi$ [8], with Ψ being a Lie-algebra valued gauge function. Therefore we find $F_{\mu\varphi} = D_\mu W$, where $W = A_\varphi - \Psi$.

The energy density of the solutions is given by the tt-component of the energy momentum tensor T_μ^ν; integration over all space yields their energy

$$E = \int 2Tr\{-F_{\mu t}F^{\mu t} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\}\sqrt{-g}d^3x. \tag{3}$$

The electric part in the above relation can be expressed as a surface integral at infinity

$$-E_e = \int Tr\{F_{\mu t}F^{\mu t}\}\sqrt{-g}d^3x = \oint_{\infty} Tr\{A_t F^{\mu t}\}dS_{\mu}. \quad (4)$$

Thus, similar to the asymptotically flat case [6], a vanishing magnitude of the electric potentials at infinity $|A_t|$ implies a purely magnetic solution. For the gauge invariant nonabelian charges we use the expressions [6, 9, 10]

$$Q_e = \frac{1}{4\pi}\oint_{\infty} |{}^*F|, \quad Q_M = \frac{1}{4\pi}\oint_{\infty} |F|, \quad (5)$$

where the vertical bars denote the Lie-algebra norm. The total angular momentum of a solution is given by

$$J = \int T^t_{\varphi}\sqrt{-g}d^3x = \int 2Tr\{F_{r\varphi}F^{rt} + F_{\theta\varphi}F^{\theta t}\}\sqrt{-g}d^3x, \quad (6)$$

and can be expressed as a surface integral at infinity [7]

$$J = \oint_{\infty} 2Tr\{WF^{\mu t}\}dS_{\mu}. \quad (7)$$

The usual ansatz used when discussing axially symmetric YM configurations in spherical coordinates is

$$A_\mu dx^\mu = \frac{1}{2e}\left[\tau^n_\phi\Big(\frac{H_1}{r}dr + (1-H_2)\,d\theta\Big) - n\Big(\tau^n_r H_3 + \tau^n_\theta\,(1-H_4)\Big)\sin\theta d\phi + (\tau^n_r H_5 + \tau^n_\theta H_6)\,dt\right], \quad (8)$$

with the Pauli matrices $\vec{\tau} = (\tau_x, \tau_y, \tau_z)$ and $\tau^n_r = \vec{\tau}\cdot(\sin\theta\cos n\phi, \sin\theta\sin n\phi, \cos\theta)$, $\tau^n_\theta = \vec{\tau}\cdot(\cos\theta\cos n\phi, \cos\theta\sin n\phi, -\sin\theta)$, $\tau^n_\phi = \vec{\tau}\cdot(-\sin n\phi, \cos n\phi, 0)$ [11]. The six gauge field functions H_i depend only on r and θ. To fix the residual abelian gauge invariance we choose the gauge condition $r\partial_r H_1 - \partial_\theta H_2 = 0$ [12, 10]. The integer n represents the winding number of the solutions. For $n = 1$ and $H_1 = H_3 = H_5 = H_6 = 0$, $H_2 = H_4 = w(r)$ the spherically symmetric magnetic ansatz of ref. [3] is recovered.

For the magnetic potentials we impose the familiar boundary conditions $H_2 = H_4 = 1,\ H_1 = H_3 = 0$ at the origin and $H_2 = H_4 = w_0,\ H_1 = H_3 = 0$ at infinity, where w_0 has an arbitrary value. Given the parity reflection symmetry, we need to consider solutions only in the region $0 \le \theta \le \pi/2$. The boundary conditions satisfied by the magnetic potentials along the axes ($\theta = 0, \pi/2$) are $H_1 = H_3 = 0,\ \partial_\theta H_2 = \partial_\theta H_4 = 0$.

For $\Lambda < 0$, there are no boundary conditions to exclude a nonabelian solution with nonzero electric potential. In [13] numerical arguments have been presented for the existence of axially symmetric YM solutions with nonvanishing magnetic and electric charges and winding number $n > 1$. These solutions generalize the $n = 1$ spherically symmetric dyons discussed in [1]. For that type of configurations, the electric potentials H_5, H_6 satisfy a set of boundary conditions inspired by the flat-space Yang-Mills-Higgs (YMH) dyons [14]. For example, we imposed at infinity $H_5 = V, H_6 = 0$, while at the origin $H_5 = H_6 = 0$. Nevertheless, we find that these dyon configurations posses a vanishing total angular momentum J [15].

However, already for a vanishing cosmological constant, the above boundary conditions does not exhaust all possibilities even for $n = 1$. The monopole-antimonopole solutions (MAP) in a YMH theory [16, 17] or the asymptotically flat rotating EYM black holes discussed in [10] are examples of nonspherically symmetric configurations with unit winding numbers. It is natural to expect that similar configurations will exist also in a AdS geometry. In what follows, we use for the electric potentials a set of boundary conditions inspired by the MAP configurations. We impose $H_5\sin\theta + H_6\cos\theta = 0,\ H_{5,r}\cos\theta - H_{6,r}\sin\theta = 0$ at the origin and $H_5 = V\cos\theta,\ H_6 = V\sin\theta$ at infinity. The boundary conditions on the symmetry axis ($\theta = 0$) are $\partial_\theta H_5 = H_6 = 0$, while for $\theta = \pi/2$ we impose $H_5 = \partial_\theta H_6 = 0$. In this letter we consider only solutions with the lowest winding number $n = 1$.

The boundary conditions for the gauge field functions at infinity imply that the solutions have a magnetic charge $Q_M = |1 - \omega_0^2|$. The value of the electric charge can also be obtained from the asymptotics of the electric potentials, since as $r \to \infty$, $H_5 \sim \cos\theta\big(V + (c_1\sin^2\theta + c_2)/r\big),\ H_6 \sim \sin\theta\big(V + (c_3\sin^2\theta + c_4)/r\big)$.

Solutions in a fixed AdS background.– Because the asymptotic structure of geometry is different, in an AdS spacetime one does not have to couple the YM system to scalar fields or gravity in order to obtain finite energy solutions. Here the cosmological constant breaks the scale invariance of pure YM theory to give finite energy solutions [3, 13]. For spherical symmetry, finite energy monopole configurations are found for only one interval of the parameter b that specifies the initial conditions at the origin. Depending on the value of b, the gauge function $\omega(r)$ is nodeless or presents one node only.

We start by presenting rotating dyons solutions of YM equations in a four-dimensional AdS spacetime, gravity being regarded as a fixed field. Although being extremely simple, nevertheless this model appears to contain all the essential features of the gravitating solutions. Also, it is much easier to solve the field equations in this case.

Subject to the above boundary conditions, we solve the YM equations numerically. The numerical calculations are performed by using the program FIDISOL [18], based on the iterative Newton-Raphson method. To map spatial infinity to a finite value, we employ the radial coordinate $\bar{r} = r/(r+c)$, where c is a properly chosen constant. As initial guess in the iteration procedure, we use the static spherically symmetric $SU(2)$ regular solutions in fixed AdS background [13]. The typical relative error for the gauge functions is estimated to be lower than 10^{-3}, while the relations (4) and (7) are verified with a very good accuracy. For all the solutions we present here we consider a cosmological constant $\Lambda = -3$. However, a similar general behavior has been found for other negative values of Λ. Similar to the spherically symmetric case [3], solutions with different values of Λ are related through a scaling transformation.

The solutions depend on two continuos parameters: the values ω_0 of the magnetic potentials $H_2,\ H_4$ at infinity and the magnitude of the electric potential at infinity V. Similar to the static case, the functions H_2 and H_4 are nodeless or present one node only, although they have a small θ dependence. For a given Λ, we found nontrivial rotating solutions for every value of ω_0. A nonvanishing V leads to rotating regular configurations, with nontrivial functions $H_1,\ H_3,\ H_5,\ H_6$. As we increase V from zero while keeping ω_0 fixed, a branch of solutions forms. This branch extends up to a maximal value of V, which depends slightly on ω_0. Along this branch, the total energy, electric charge, electric part of energy and the absolute value of the angular momentum increase continuously with V. Depending on V, the energy of a rotating solution can be several order of magnitude greater than the energy of the corresponding monopole solution. We find that both E_e/E and J/E^2 tend to constant values as V is increased. At the same time, the numerical errors start to increase, we obtain large values for both Q_e and E, and for some V_{max} the numerical iterations fail to converge. In this limit, the total energy and the electric charge diverge, while the magnetic charge takes a finite value. An accurate value of V_{max} is rather difficult to obtain, especially for large values of ω_0. Alternatively, we may keep fixed the magnitude of the electric potential at infinity and vary the parameter ω_0. In Figure 1 we present the properties of typical branches of solutions for a fixed value of ω_0 (Figure 1a) and for a fixed V (Figure 1b).

For all configurations, the energy density $\epsilon = -T_t^t$ of the solutions has a strong peak along the ρ axis, and it decreases monotonically along the symmetry axis, without being possible to distinguish any individual component. Equal density contours reveal a torus-like shape of the solutions. Dyon solutions are found in a good portion of $Q_M - Q_e$ plane. There are also solutions where $Q_M = 0$ and $Q_e \neq 0$. A vanishing Q_e implies a nonrotating, purely magnetic configuration. However, we find dyon solutions with vanishing total angular momentum ($J = 0$ for some $\omega_0 < 1$) which are not static (locally $T_\varphi^t \neq 0$) [19]. Resembling the $\Lambda = 0$ MAP solutions [16, 17], the modulus of the electric potential $|A_t|$ possesses two zeros at $\pm z_0$, on the z axis.

Inclusion of gravity.– We use a metric form inspired by the asymptotically flat ansatz [10], which satisfies also the circularity condition [20]

$$ds^2 = \frac{m}{f}\left(\frac{dr^2}{1-\frac{\Lambda}{3}r^2} + r^2 d\theta^2\right) + \frac{l}{f} r^2 \sin^2\theta (d\phi + \frac{\omega}{r}dt)^2 - f(1 - \frac{\Lambda}{3}r^2)dt^2, \qquad (9)$$

with f, l, m and ω being functions of r and θ. To obtain asymptotically AdS regular solutions with finite energy density, the metric functions have to satisfy the boundary conditions $f = m = l = 1,\ \omega = 0$ at infinity, and $\partial_r f = \partial_r m = \partial_r l = \omega = 0$ at the origin. The boundary conditions on the symmetry axis are $\partial_\theta f = \partial_\theta m = \partial_\theta l = \partial_\theta \omega = 0$, and agree with the boundary conditions on the $\theta = \pi/2$ axis. We remove the dependence on the coupling constants G and e from the differential equations by changing to dimensionless

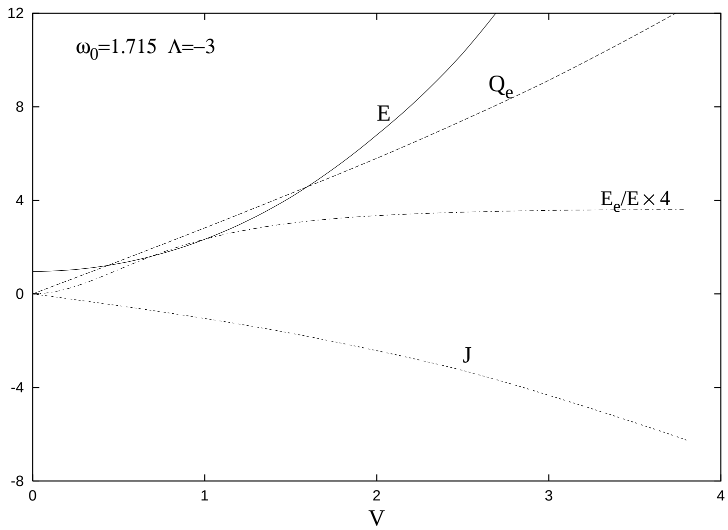

Figure 1a.

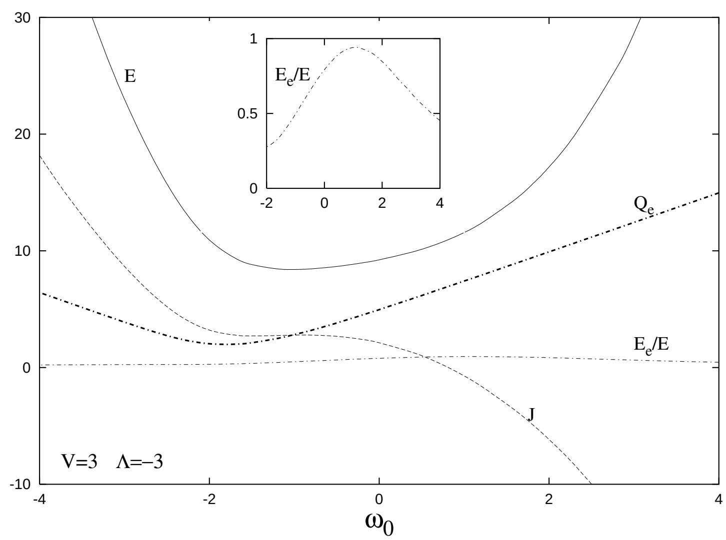

Figure 1b.

Figure 1. The energy E and the angular momentum J (in units $4\pi/e^2$) of non-Abelian regular solutions in fixed AdS background with $\Lambda = -3$ are shown as a function on the parameter V (Figure 1a, for $\omega_0 = 1.715$) and the parameter ω_0 (Figure 1b, $V = 3$). Also shown are the electric charge Q_e and the ratio E_e/E.

radial coordinate $r \to (\sqrt{4\pi G}/e)r$ and also $\Lambda \to (e^2/4\pi G)\Lambda$. The dimensionless mass is obtained by rescaling $M \to (eG/\sqrt{4\pi G})M$. The ADM mass and angular momentum can be derived from the asymptotics of the metric functions [21].

To solve the EYM equations with a negative cosmological constant we employ the same numerical algorithm as for the YM solutions in fixed AdS background. Similar to other gravitating nonabelian configurations with axial symmetry [12, 13], we use in the numerical procedure a suitable combination of the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{10}$$

such that the diferential equations for metric variables (f, m, l, ω) are diagonal in the second derivatives with respect to r.

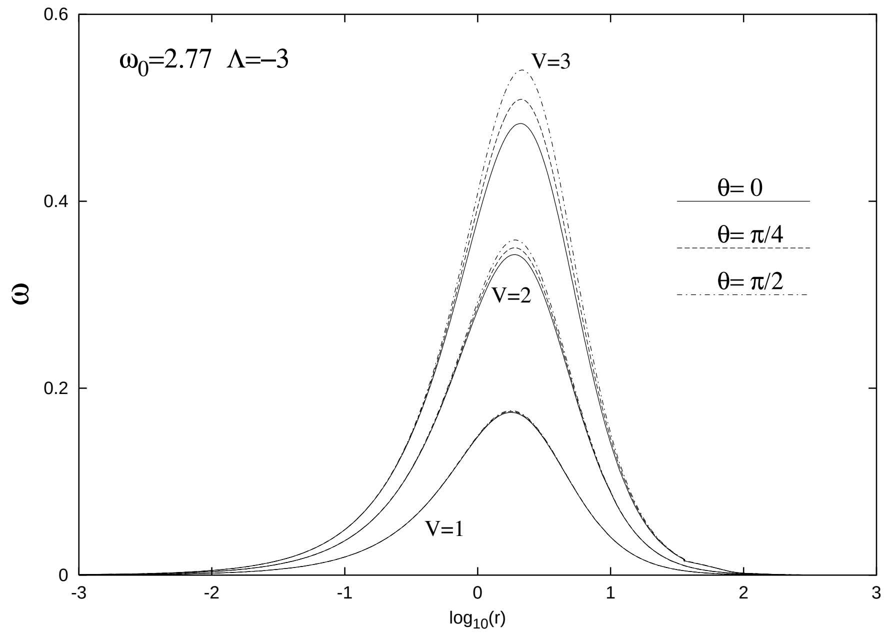

Figure 2. The metric function $\omega(r,\theta)$ is shown as a function of the radial coordinate r for three fundamental branch solutions with the same ω_0 and different values of the electric potential magnitude at infinity.

In the spherically symmetric case, gravitating regular solutions are obtained for a finite number of compact intervals of the parameter b. The properties of these solutions depend essentially on the value of Λ. The allowed values of b for YM solutions in an AdS background with a small $|\Lambda|$ correspond approximately to the lower branch when coupling to gravity.

We start with a $n=1$ purely magnetic EYM solution obtained in isotropic coordinates [13] as initial guess and increase the value of V slowly (for a fixed ω_0). We constructed in this way gravitating dyon solutions with nonvanishing angular momentum for a range of $(\Lambda,\ \omega_0,\ V)$. The branch structure of the spherically symmetric solutions is preserved in the presence of rotation. When gravity is coupled to YM theory, a branch of gravitating dyon solutions emerges smoothly from the dyon solutions of AdS space and extending up to some maximal value of V beyond which gravity becomes too strong for regular dyons to persist. This value is always smaller than the corresponding value in a fixed AdS background. These lowest branch solutions are of particular interest because some of them are likely to be stable against linear perturbations. For small values of the cosmological constant (typically $|\Lambda| < 10^{-1}$), the basic properties of these fundamental gravitating solutions are similar to those of the solutions in the absence of gravity. The general picture presented in Figure 1 provides a good qualitative description in this case too. We notice that the value at the origin of the metric function f decreases with increasing V and tends to zero as V approaches the critical value V_{max}, corresponding to the formation of a horizon.

Also, for the studied configurations we find that the total mass of a gravitating solution has a smaller value than the mass of the corresponding solution in a fixed AdS background. This inequality is of course in accord with our intuition that gravity tends to reduce the mass. A similar property has been noticed for monopole solutions in a spontaneously broken gauge theory [22]. As expected, the gauge functions H_i look very similar to those of the corresponding (pure-) YM solutions. With increasing V, the dyon becomes more and more deformed. The form of the non-diagonal metric function ω clearly demonstrates the differential rotation of different portions of these torus-like configurations (see Figure 2).

Beside these fundamental gravitating solutions, EYM theory with $\Lambda < 0$ may possess also excited solutions not presented in a fixed AdS background. These solutions are obtained by starting with an excited-sherically symmetric EYM monopole solution [1], and slowly increasing the value of V. We motice the existence of higher node configurations in this case, the maximal node number depending on Λ.

For higher branches solutions, we find the same general picture. The mass, angular momentum and electric charges increase with V and we find again a maximal value for the magnitude of the electric potential

at infinity. Again, as the critical value of V is approached, the metric function f develops a zero at the origin, corresponding to the formation of a horizon. Also, we find that the metric functions $(f,\ l,\ m)$ of the excited EYM solutions are considerably smaller at the origin, and the gauge field functions have their peaks and nodes shifted inwards, as compared to the corresponding first branch solutions. Higher branches rotating solutions are more difficult to obtain. The numerical error for gravitating configurations is estimated to be on the order of 10^{-3} for first branch solutions and 10^{-2} in rest. This error depends also on the values of ω_0 and V.

However, for large values of $|\Lambda|$, the picture presented in Figure 1b is no longer valid even for fundamental gravitating solutions. For example, we find in this case a minimal allowed value for the parameter ω_0 and the absence of higher branches solutions. These differences emerge from the distinctions existing already in the spherically symmetric case (with $V = 0$), distinctions not yet discussed in the literature.

More details on these rotating regular solutions will be given elsewhere.

Further remarks.– We found that the gravitating rotating solitons depend nontrivially on the value of the cosmological constant Λ. As discussed in [1, 3], the higher branches of gravitating monopoles with $\Lambda < 0$ are related to Bartnik-McKinnon (BM) solutions [23]. Naively, one may expect that the corresponding rotating dyon configurations are also related to rotating generalizations of BM solutions. In this limit the asymptotics of the electric potential simplifies: $c_1 = c_3 = 0,\ c_2 = c_4$ and therefore $J/Q_e = const.$ as predicted in [4]. Nevertheless, we find numerically that, in the limit $\Lambda \to 0$, the maximal value of V also tends to zero and, for higher branches, we recover the nonrotating BM solution (while the fundamental branch approaches the vacuum solution). Although further research is clearly necessary, it seems that for an axially symmetric regular configuration with $\Lambda = 0$, similar to the spherically symmetric case, the electric part of the axially symmetric gauge fields is forbidden if the ADM mass is to remain finite. If $V \neq 0$, the A_t^a components of the gauge field act like an isotriplet Higgs field with negative metric, and by themselves would cause the other components of the gauge field to oscillate as $r \to \infty$ [7].

A simple way to remedy this fact is to include a triplet Higgs field in the theory. The Higgs field functions will satisfy a set of boundary conditions similar to the corresponding electric YM potentials. However, the magnitude of the Higgs field at infinity should be greater that V. This configuration will correspond to an asymptotically flat, rotating MAP solution first predicted in [14], with a vanishing net magnetic charge but with a nonzero electric charge. In this case too, the electric charge and the angular momentum are not independent quantities any longer [7].

Also, the existence of other branches of asymptotically AdS rotating regular solutions, not necessarily connected to the static solutions might be possible.

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